HÖGSKOLAN I GÄVLE

Review of Automatic Control

Frequency response

Per Mattsson

per.mattsson@hig.se

Introduction

- We have seen how the system reacts to a step in the input.

per.mattsson@hig.se

Introduction

- We have seen how the system reacts to a step in the input.
- We will now study how the system reacts to signals of different frequencies.

per.mattsson@hig.se

Introduction

- We have seen how the system reacts to a step in the input.
- We will now study how the system reacts to signals of different frequencies.
- Many signals can be written as a sum of sinusoids of different frequencies (Fourier series).

per.mattsson@hig.se

Introduction

- We have seen how the system reacts to a step in the input.
- We will now study how the system reacts to signals of different frequencies.
- Many signals can be written as a sum of sinusoids of different frequencies (Fourier series).
- For a linear system the output is then a linear combination of the frequency responses for each frequency in the input.

per.mattsson@hig.se

Frequency response

Consider the sinusoid input

$$u(t) = K \sin(\omega t + \phi).$$

per.mattsson@hig.se

Frequency response

Consider the sinusoid input

$$u(t) = K\sin(\omega t + \phi).$$

If the system $G(s)$ is stable then, after the transients have died out,

$$y(t) = |G(i\omega)|K\sin(\omega t + \phi + \arg(G(i\omega))).$$

per.mattsson@hig.se

Frequency response

Consider the sinusoid input

$$u(t) = K\sin(\omega t + \phi).$$

If the system $G(s)$ is stable then, after the transients have died out,

$$y(t) = |G(i\omega)|K\sin(\omega t + \phi + \arg(G(i\omega))).$$

- **Amplitude gain:** $|G(i\omega)|$.

per.mattsson@hig.se

Frequency response

Consider the sinusoid input

$$u(t) = K\sin(\omega t + \phi).$$

If the system $G(s)$ is stable then, after the transients have died out,

$$y(t) = |G(i\omega)|K\sin(\omega t + \phi + \arg(G(i\omega))).$$

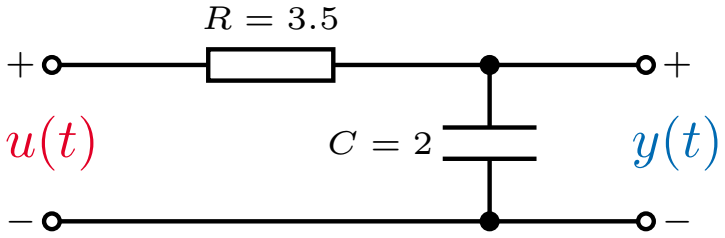
- **Amplitude gain:** $|G(i\omega)|$.
- **Phase shift:** $\arg(G(i\omega))$.

per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$

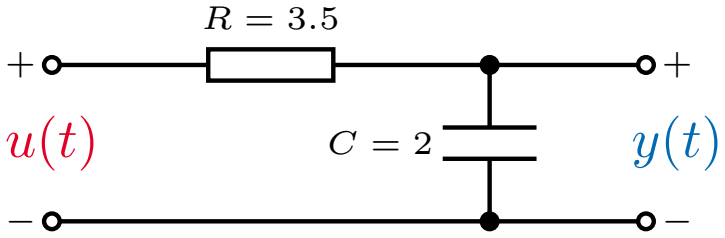


per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



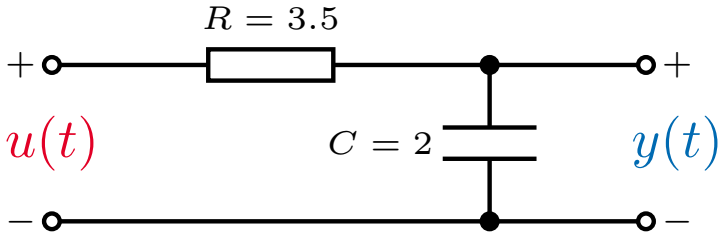
- **Amplitude gain:** $|G(i\omega)|$

per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



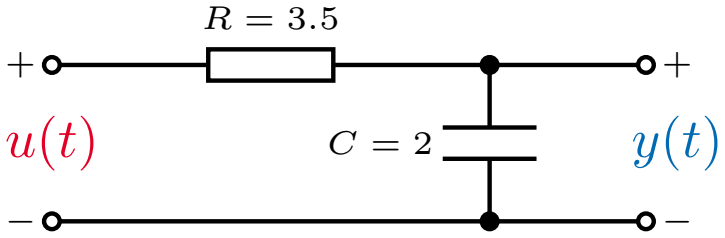
- **Amplitude gain:** $|G(i\omega)| = \dfrac{1}{\sqrt{1+49\omega^2}}.$

per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



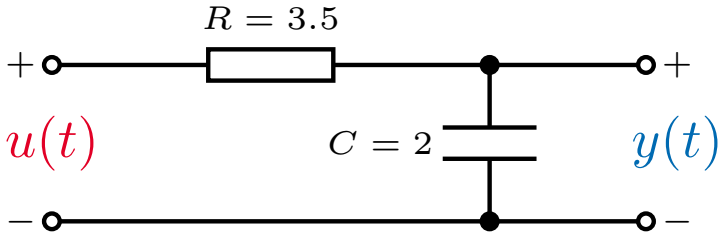
- **Amplitude gain:** $|G(i\omega)| = \dfrac{1}{\sqrt{1+49\omega^2}}.$
- **Phase shift:** $\arg(G(i\omega)) =$

per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



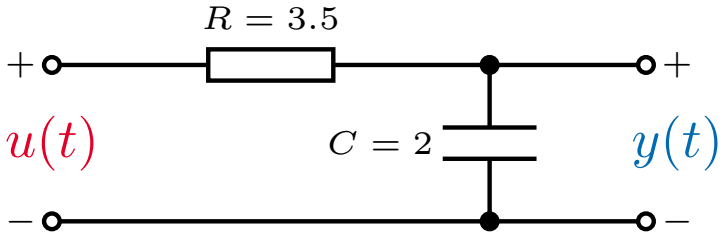
- **Amplitude gain:** $|G(i\omega)| = \dfrac{1}{\sqrt{1+49\omega^2}}.$
- **Phase shift:** $\arg(G(i\omega)) = -\arctan(7\omega).$

per.mattsson@hig.se

Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



- **Amplitude gain:** $|G(i\omega)| = \dfrac{1}{\sqrt{1+49\omega^2}}.$
- **Phase shift:** $\arg(G(i\omega)) = -\arctan(7\omega).$

The input $u(t) = \sin(\omega t)$ thus give (after the transients have died out)

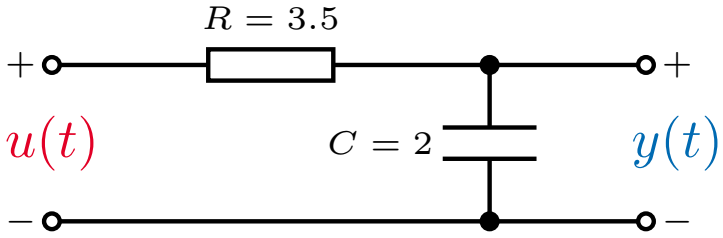
$$y(t) = \frac{1}{\sqrt{1+49\omega^2}} \sin(\omega t - \arctan(7\omega)).$$

per.mattsson@hig.se

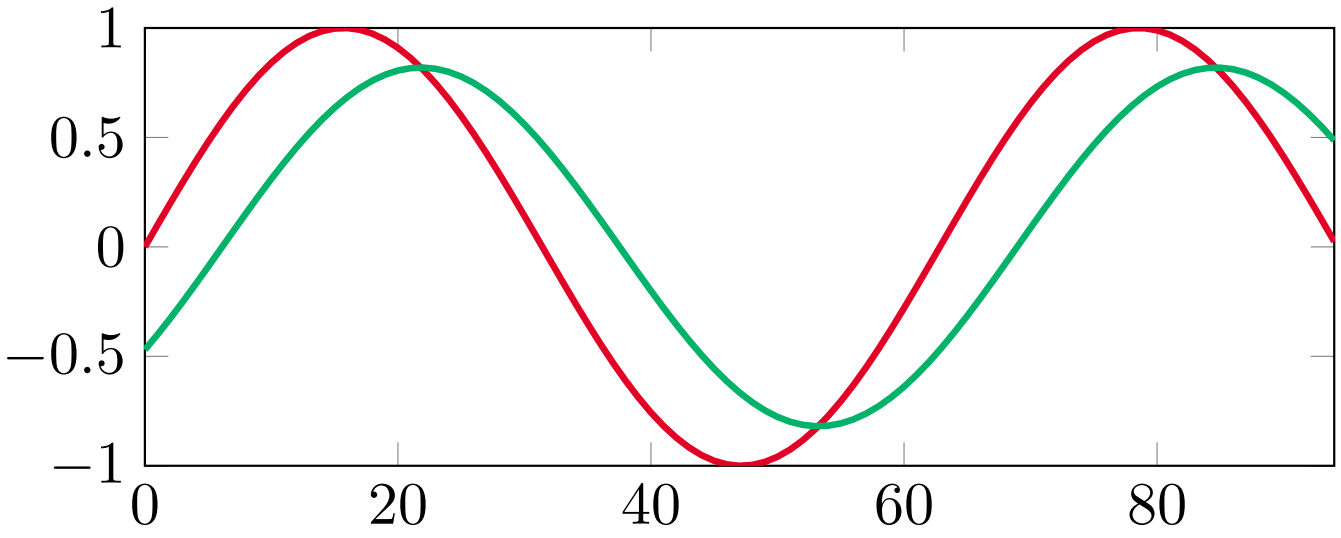
Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



$$\omega = 0.1,\ |G(j\omega)| = 0.82,\ \arg(G(j\omega)) = -0.61\,\mathtt{rad}$$



Input: $u(t) = \sin(0.1t)$.

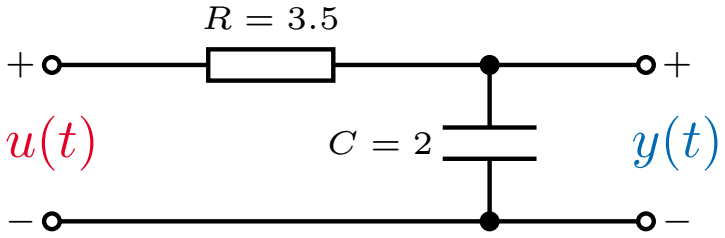
After transients: $y(t) = 0.82\sin(\omega t - 0.61)$.

per.mattsson@hig.se

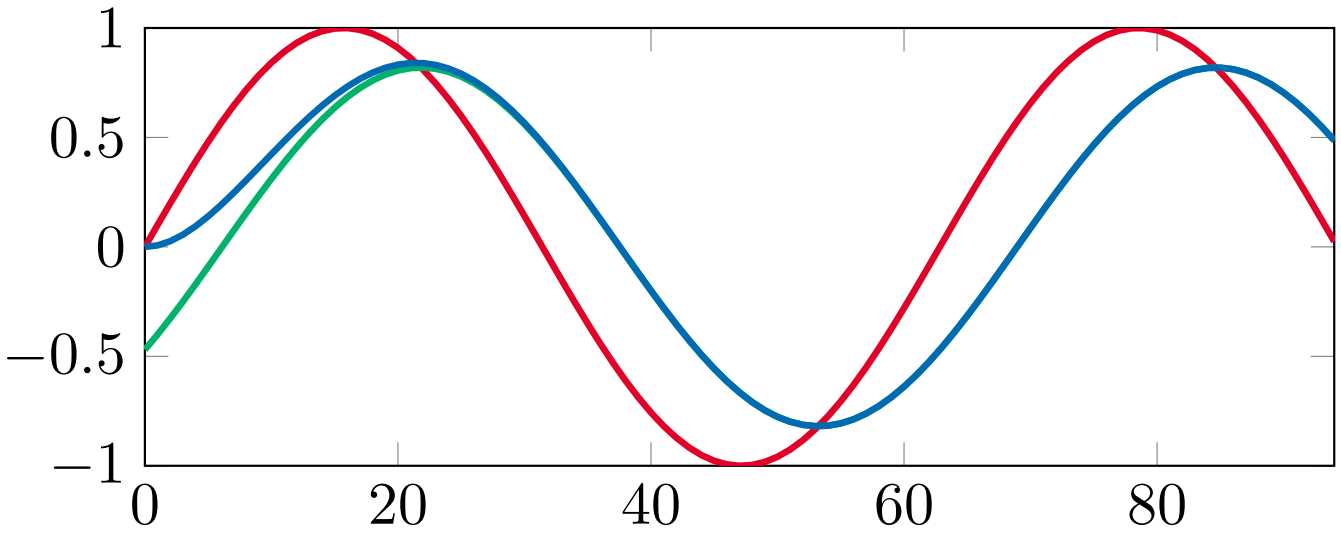
Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



$$\omega = 0.1,\ |G(j\omega)| = 0.82,\ \arg(G(j\omega)) = -0.61\,\mathtt{rad}$$



Input: $u(t) = \sin(0.1t)$.

After transients: $y(t) = 0.82\sin(\omega t - 0.61)$.

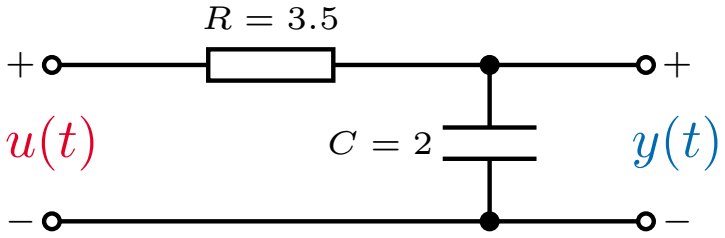
Actual output: $y(t)$ (at rest at $t = 0$).

per.mattsson@hig.se

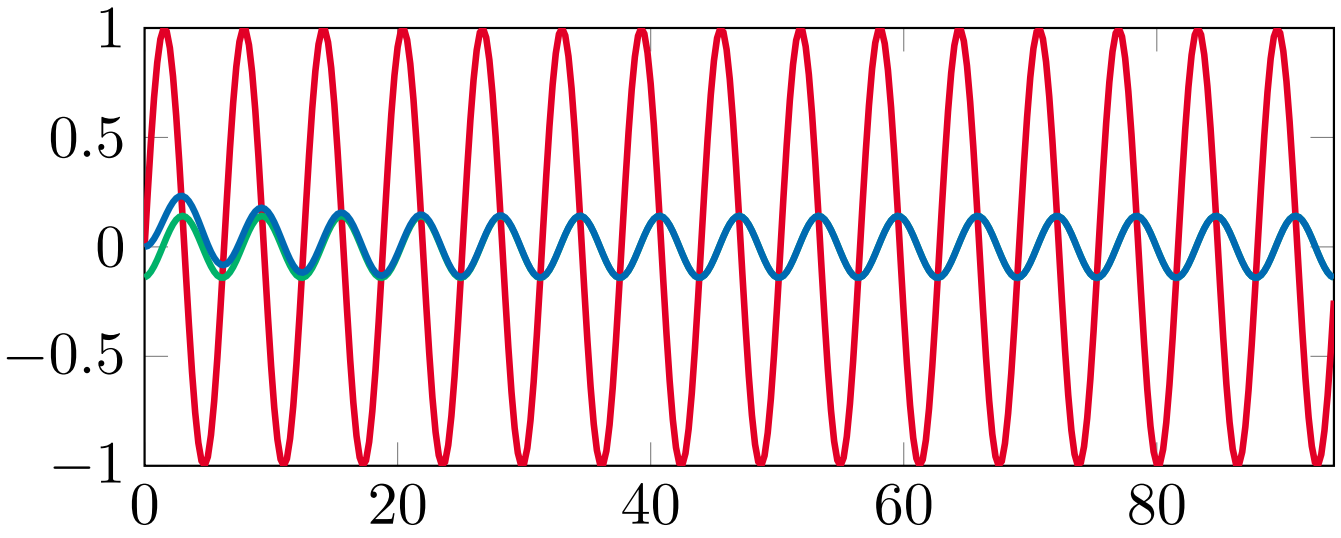
Example: Low pass filter

Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+7s}.$$



$$\omega = 1,\ |G(j\omega)| = 0.14,\ \arg(G(j\omega)) = -1.43\,\mathtt{rad}$$



Input: $u(t) = \sin(t)$.

After transients: $y(t) = 0.14\sin(t - 1.43)$.

Actual output: $y(t)$ (at rest at $t = 0$).

per.mattsson@hig.se

Bode plots

- The amplitude gain $|G(i\omega)|$ and the phase shift $\arg(G(i\omega))$ shows how the system $G(s)$ reacts to different frequencies ω in the input.

per.mattsson@hig.se

Bode plots

- The amplitude gain $|G(i\omega)|$ and the phase shift $\arg(G(i\omega))$ shows how the system $G(s)$ reacts to different frequencies ω in the input.
- It is useful by plotting them as a function of ω.

per.mattsson@hig.se

Bode plots

- The amplitude gain $|G(i\omega)|$ and the phase shift $\arg(G(i\omega))$ shows how the system $G(s)$ reacts to different frequencies ω in the input.
- It is useful by plotting them as a function of ω.
- Such plots are called Bode plots, after the American researcher Hendrik Wade Bode, who originally started using such plots in the 1930s.

per.mattsson@hig.se

Bode plots

- The amplitude gain $|G(i\omega)|$ and the phase shift $\arg(G(i\omega))$ shows how the system $G(s)$ reacts to different frequencies ω in the input.
- It is useful by plotting them as a function of ω.
- Such plots are called Bode plots, after the American researcher Hendrik Wade Bode, who originally started using such plots in the 1930s.
- A typical Bode plots:
  - **Frequency:** Often in radians/s, in logarithmic scale.
  - **Amplitude gain:** Often plotted in decibel,

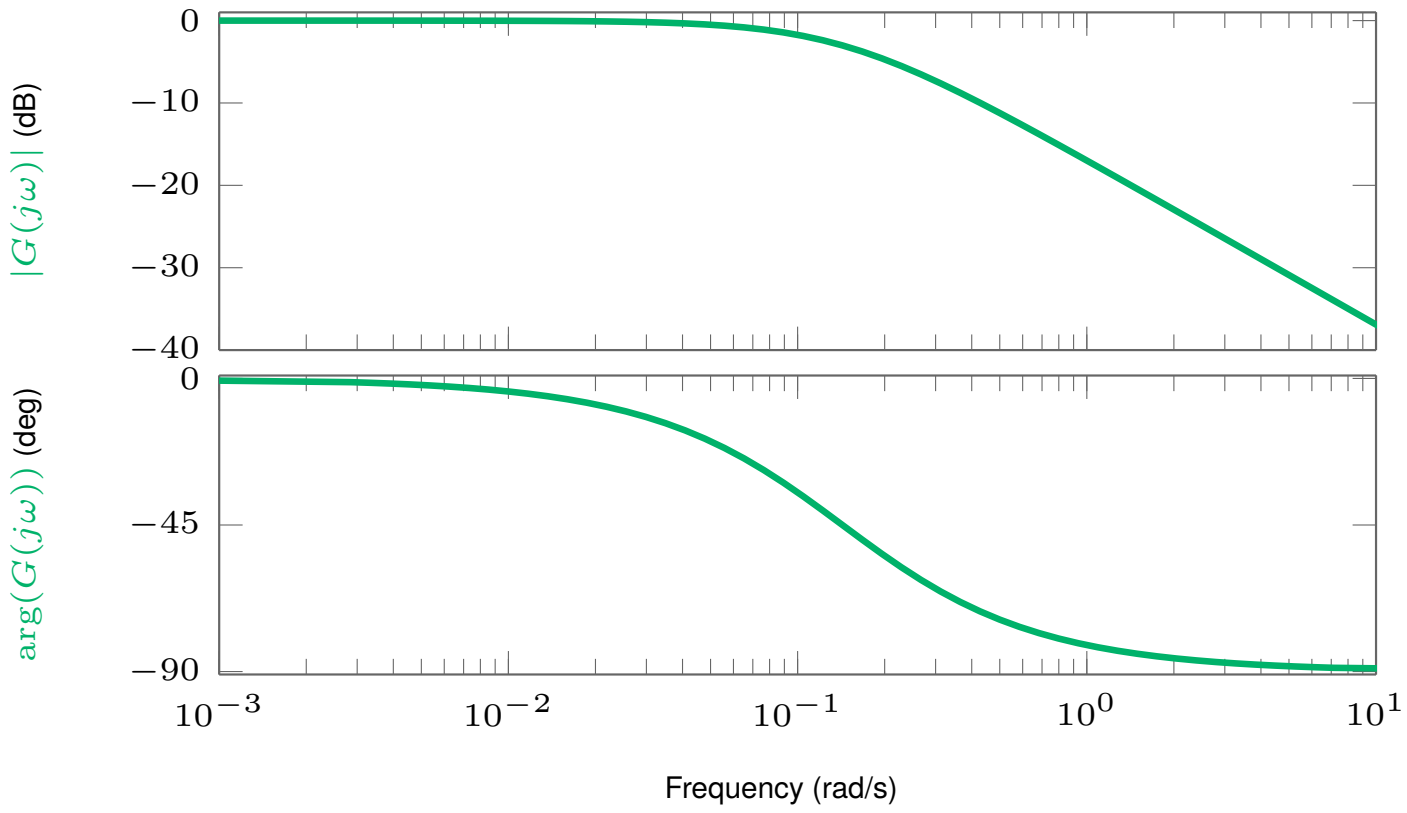
$$|G(i\omega)|_{dB} = 20\log_{10}|G(i\omega)|.$$

  - **Phase shift:** Often plotted in linear scale, with degrees as the unit.

per.mattsson@hig.se

Example

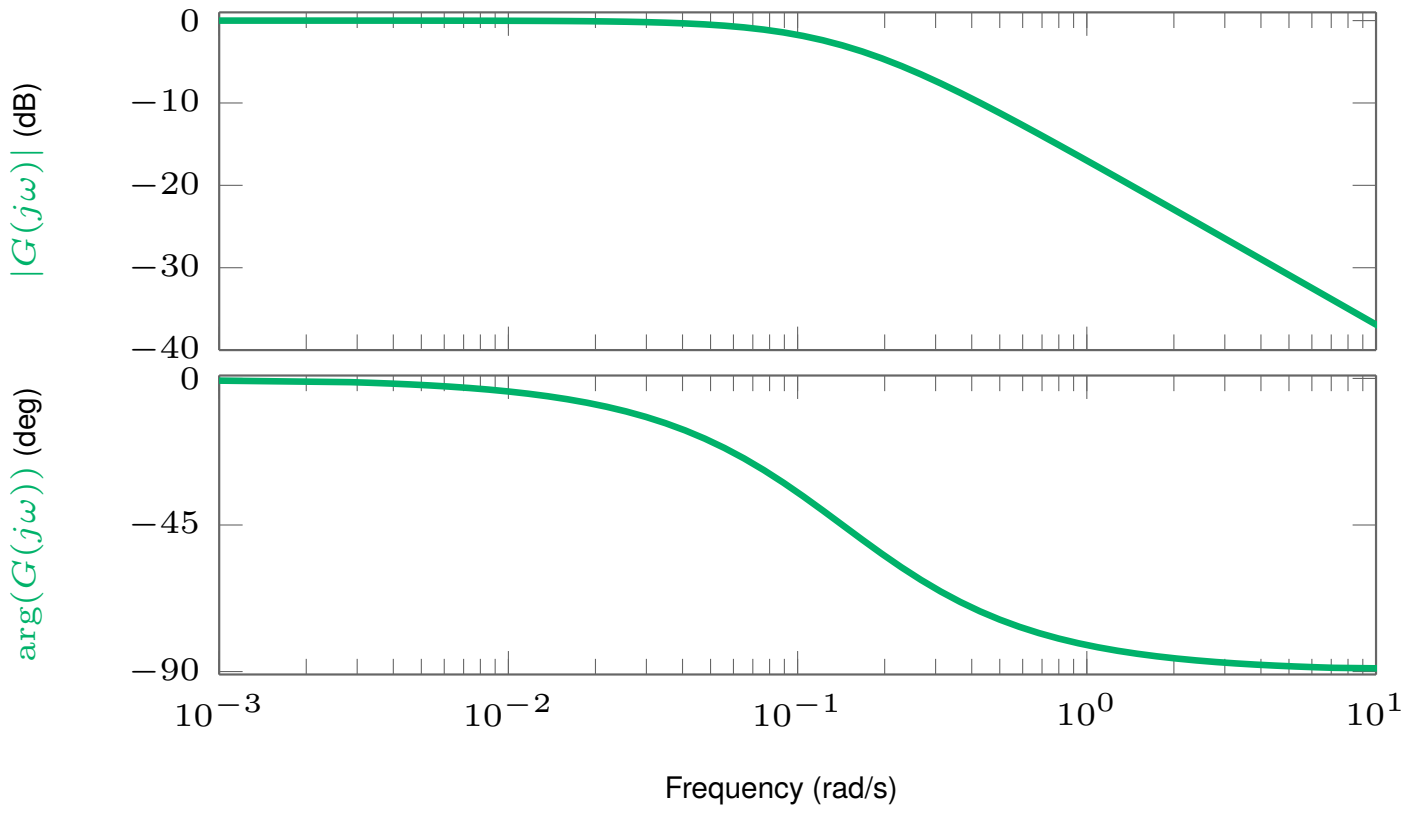
$$G(s) = \frac{1}{1+7s}, \quad |G(i\omega)| = \frac{1}{\sqrt{1+49\omega^2}}, \quad \arg(G(i\omega)) = -\arctan(7\omega).$$



per.mattsson@hig.se

Example

$$G(s) = \frac{1}{1+7s}, \quad |G(i\omega)| = \frac{1}{\sqrt{1+49\omega^2}}, \quad \arg(G(i\omega)) = -\arctan(7\omega).$$

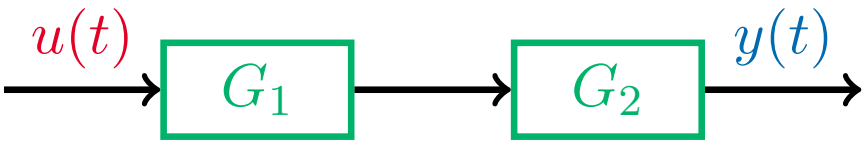


MATLAB:

```
» G = tf( [1] , [7 1])
» bode(G)
```


per.mattsson@hig.se

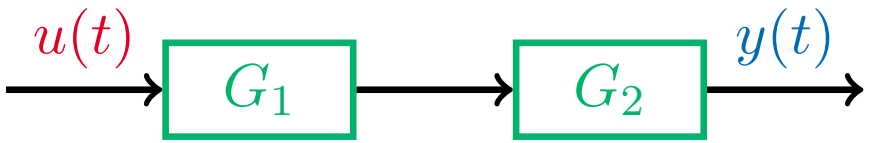
Bode plots for systems connected in series



$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

per.mattsson@hig.se

Bode plots for systems connected in series



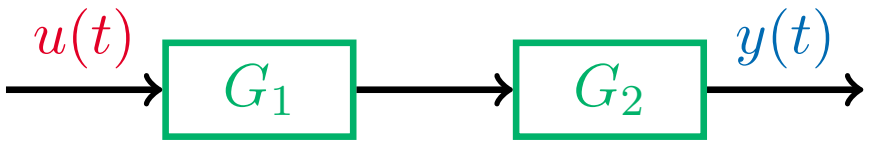
$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)|$$

per.mattsson@hig.se

Bode plots for systems connected in series



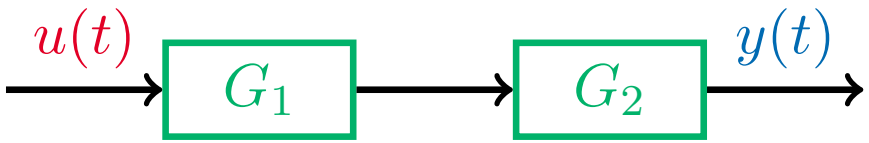
$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)| = |G_1(i\omega)||G_2(i\omega)|.$$

per.mattsson@hig.se

Bode plots for systems connected in series



$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

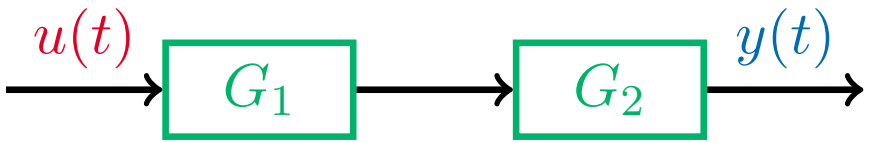
$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)| = |G_1(i\omega)||G_2(i\omega)|.$$

In decibel we thus get

$$|G(i\omega)|_{dB} = 20\log_{10}(|G_1(i\omega)|) + 20\log_{10}(|G_2(i\omega)|)$$

per.mattsson@hig.se

Bode plots for systems connected in series



$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

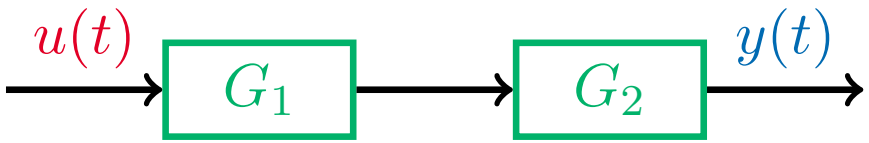
$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)| = |G_1(i\omega)||G_2(i\omega)|.$$

In decibel we thus get

$$\begin{aligned} |G(i\omega)|_{dB} &= 20\log_{10}(|G_1(i\omega)|) + 20\log_{10}(|G_2(i\omega)|) \\ &= |G_1(i\omega)|_{dB} + |G_2(i\omega)|_{dB}. \end{aligned}$$

per.mattsson@hig.se

Bode plots for systems connected in series



$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)| = |G_1(i\omega)||G_2(i\omega)|.$$

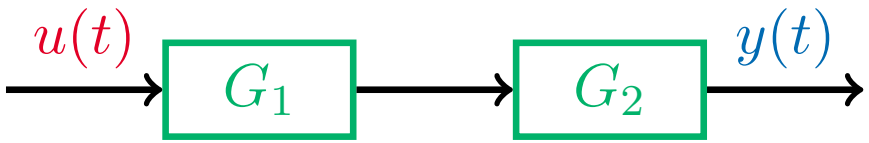
  In decibel we thus get

$$\begin{aligned} |G(i\omega)|_{dB} &= 20\log_{10}(|G_1(i\omega)|) + 20\log_{10}(|G_2(i\omega)|) \\ &= |G_1(i\omega)|_{dB} + |G_2(i\omega)|_{dB}. \end{aligned}$$

- **Phase shift:** $\arg(G(i\omega)) = \arg(G_1(i\omega)) + \arg(G_2(i\omega))$

per.mattsson@hig.se

Bode plots for systems connected in series



$$G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

- **Amplitude gain:**

$$|G(i\omega)| = |G_1(i\omega)G_2(i\omega)| = |G_1(i\omega)||G_2(i\omega)|.$$

  In decibel we thus get

$$\begin{aligned} |G(i\omega)|_{dB} &= 20\log_{10}(|G_1(i\omega)|) + 20\log_{10}(|G_2(i\omega)|) \\ &= |G_1(i\omega)|_{dB} + |G_2(i\omega)|_{dB}. \end{aligned}$$

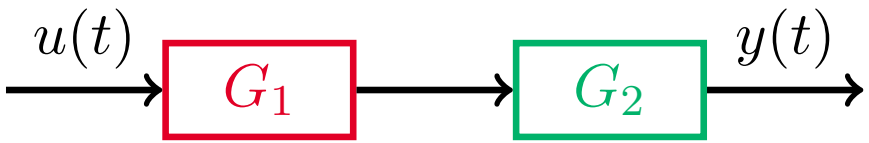
- **Phase shift:** $\arg(G(i\omega)) = \arg(G_1(i\omega)) + \arg(G_2(i\omega))$

The Bode plot for $G(s)$ is given by the adding up the amplitude and phase curves for $G_1(s)$ och $G_2(s)$!

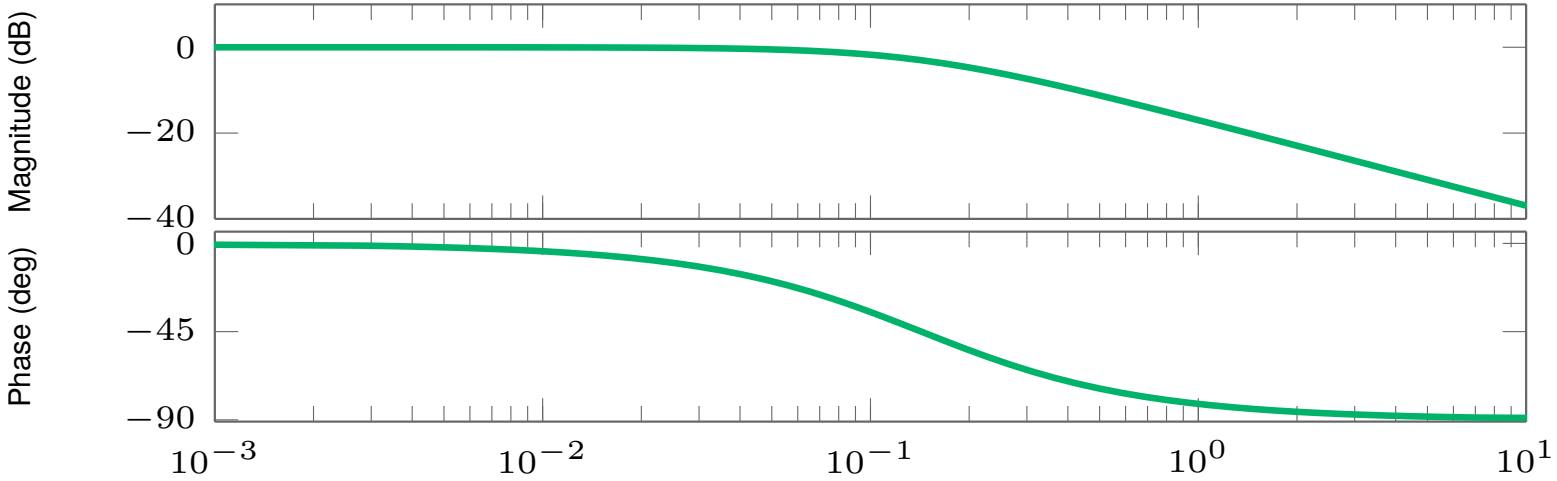
per.mattsson@hig.se

Bode plots for systems connected in series

Example



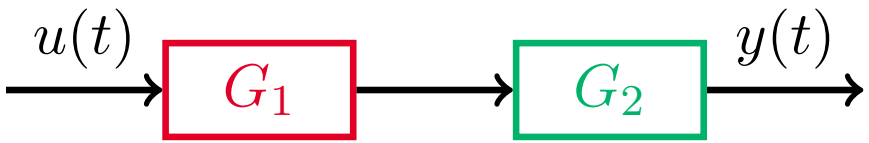
If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?



per.mattsson@hig.se

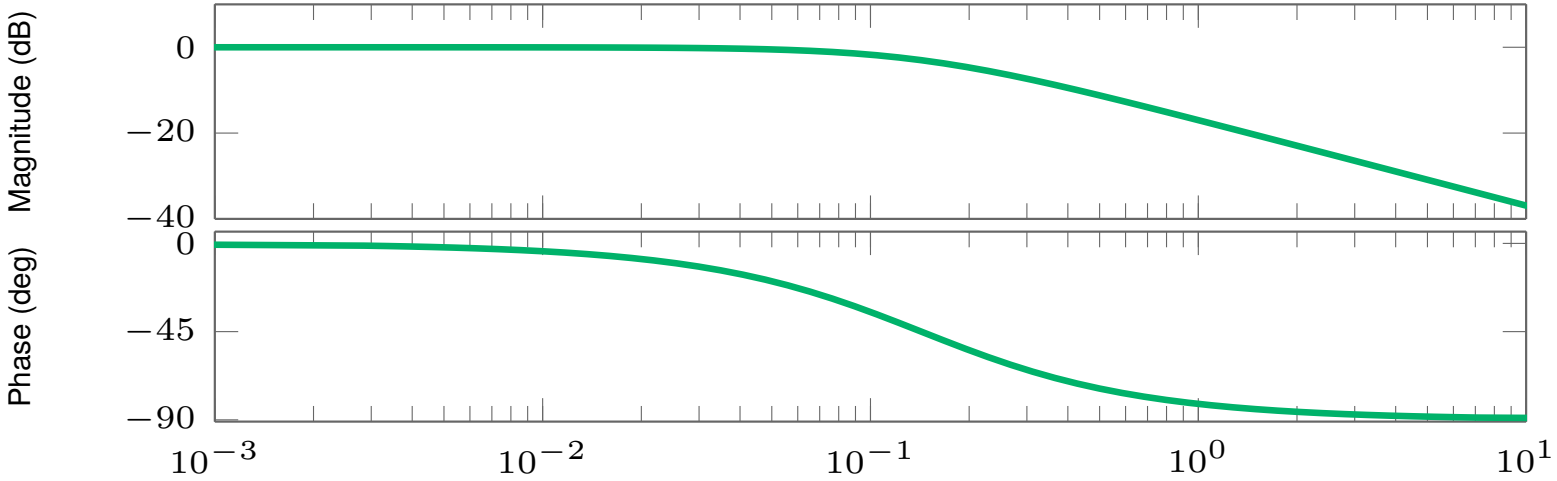
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

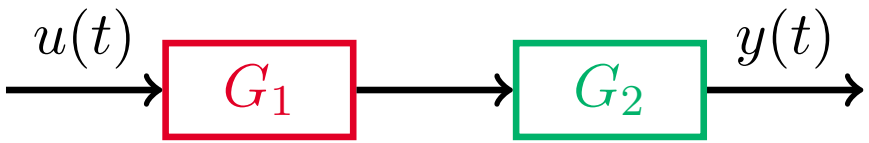
- **Amplitude gain:** $|G_1(i\omega)|_{dB} =$



per.mattsson@hig.se

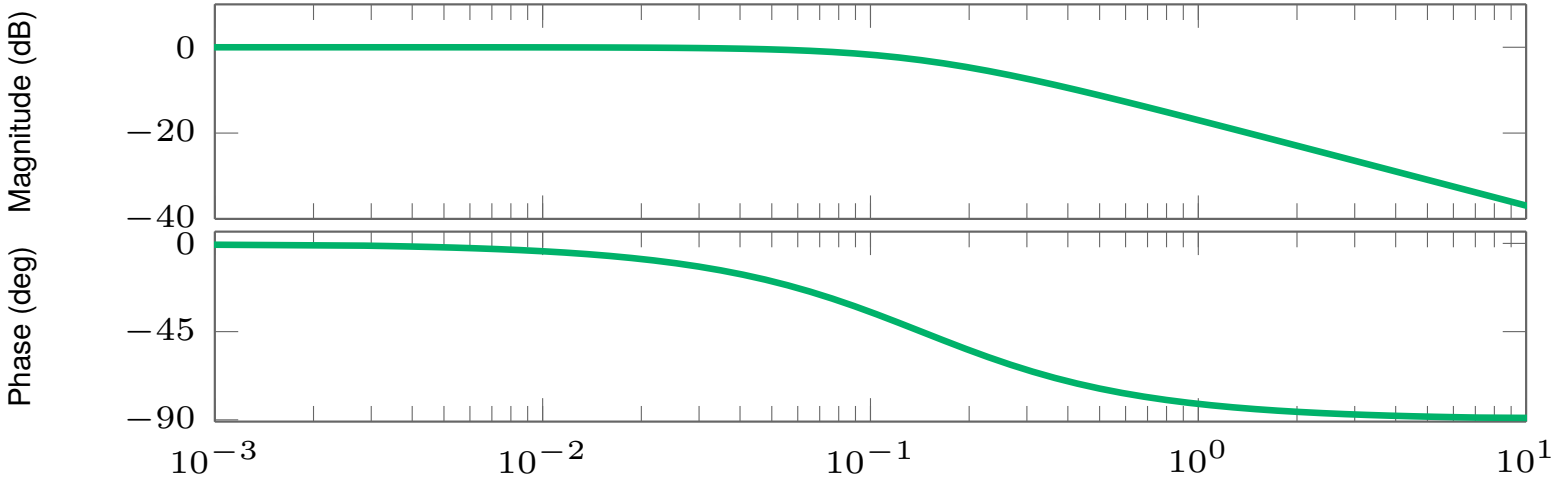
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

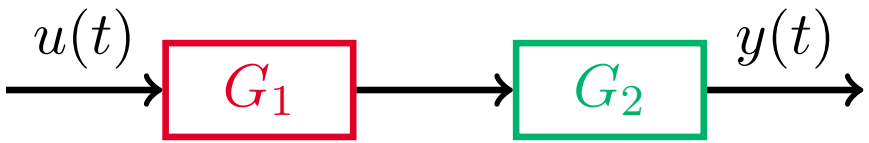
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB.$



per.mattsson@hig.se

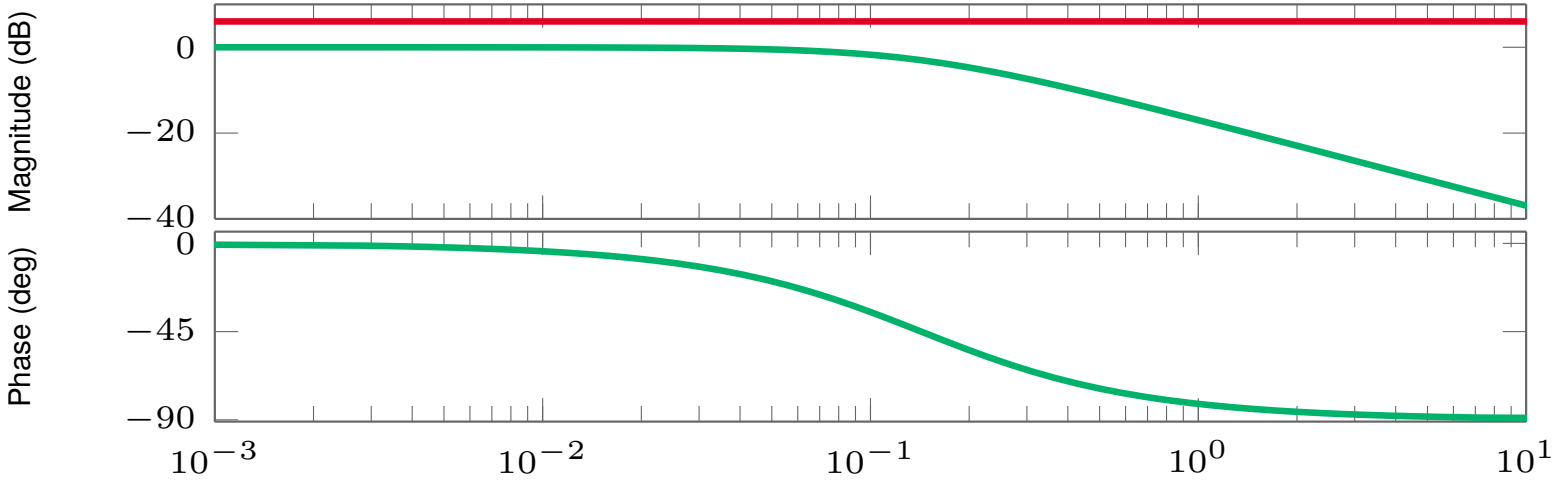
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

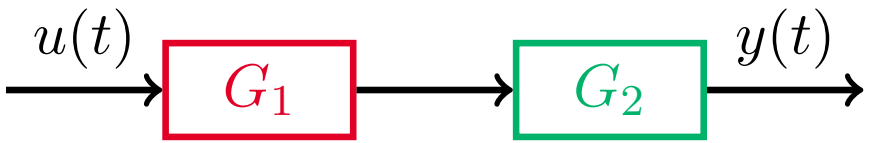
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB.$



per.mattsson@hig.se

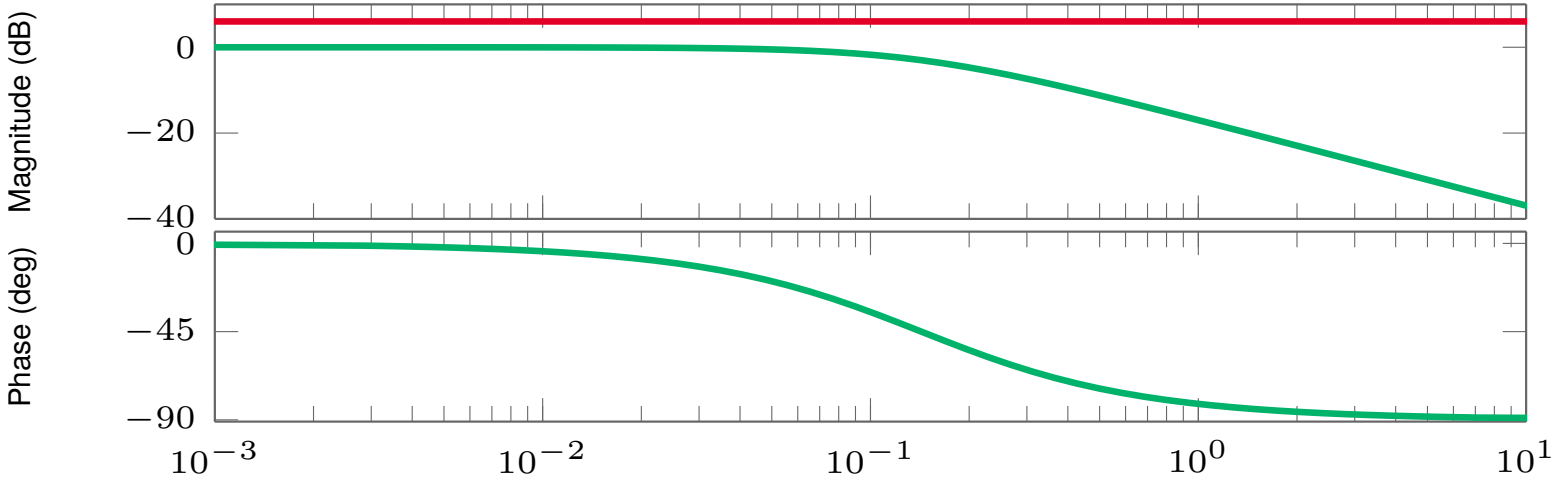
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

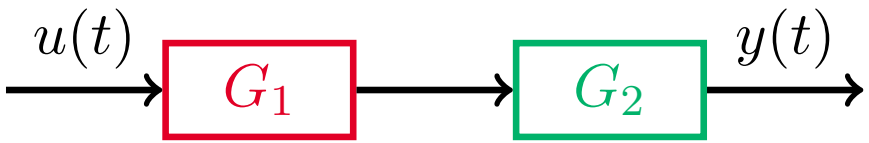
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB.$
- **Phase shift:** $\arg(G_1(i\omega)) = \arg(2) =$



per.mattsson@hig.se

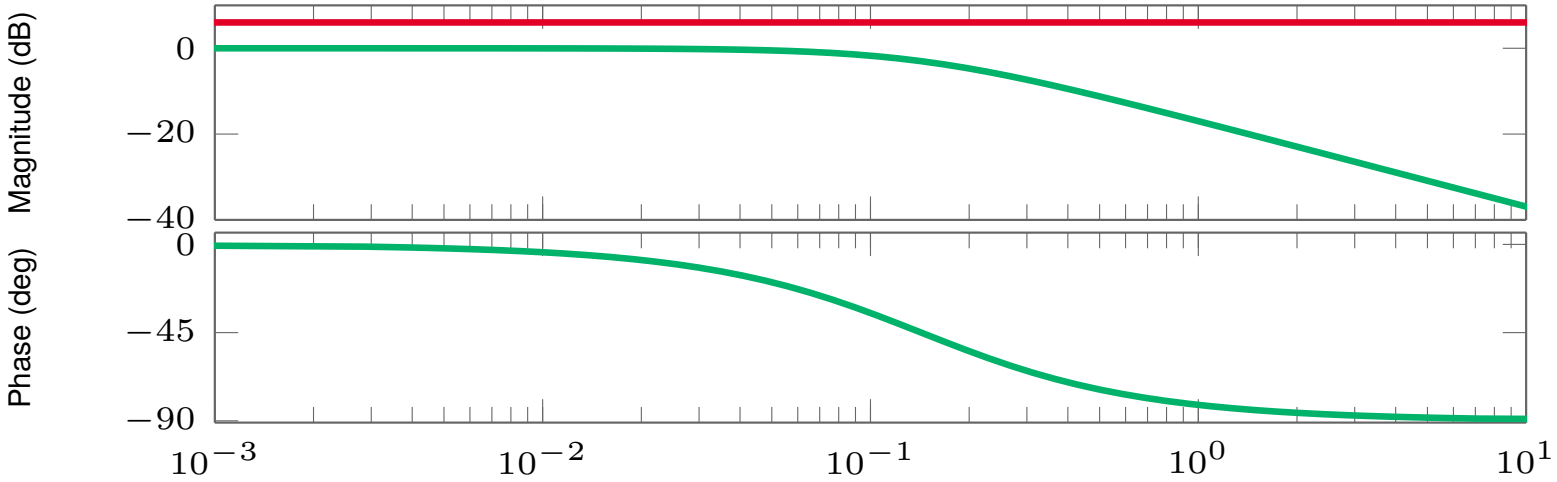
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

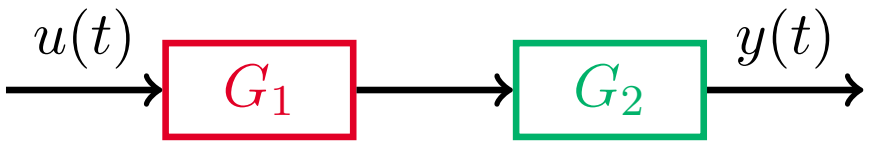
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB$.
- **Phase shift:** $\arg(G_1(i\omega)) = \arg(2) = 0$.



per.mattsson@hig.se

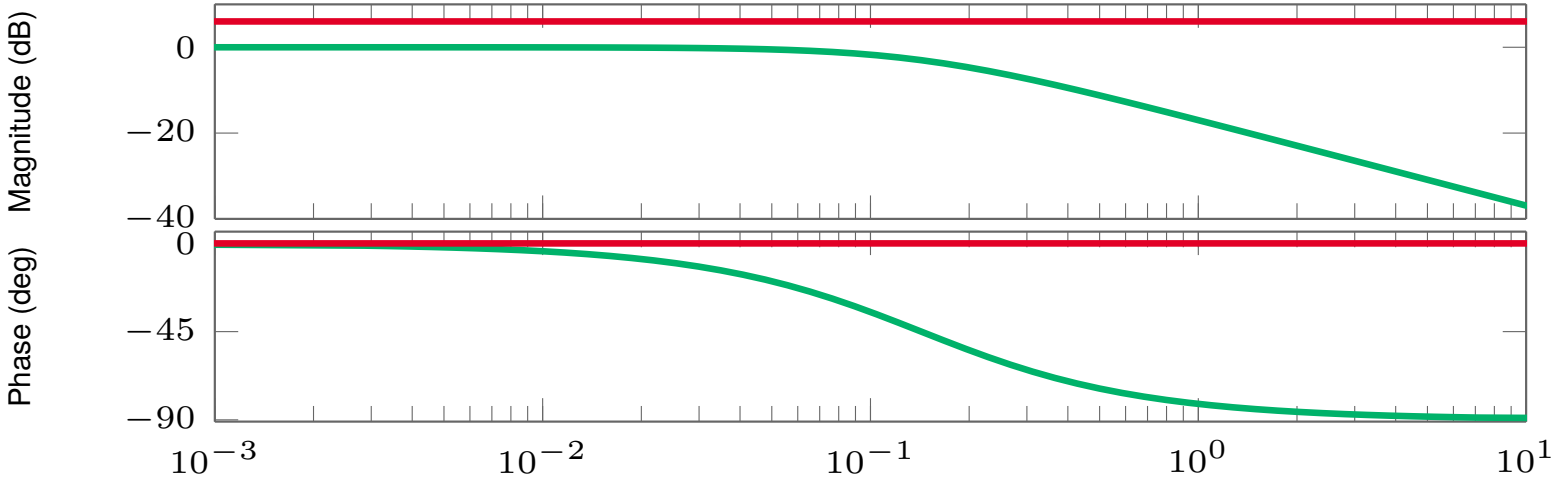
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

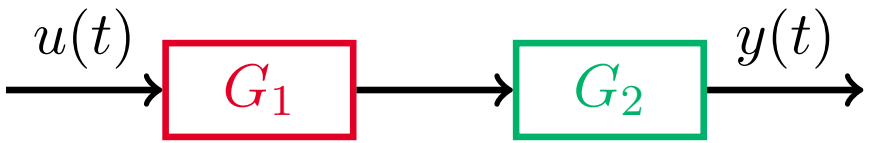
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB$.
- **Phase shift:** $\arg(G_1(i\omega)) = \arg(2) = 0$.



per.mattsson@hig.se

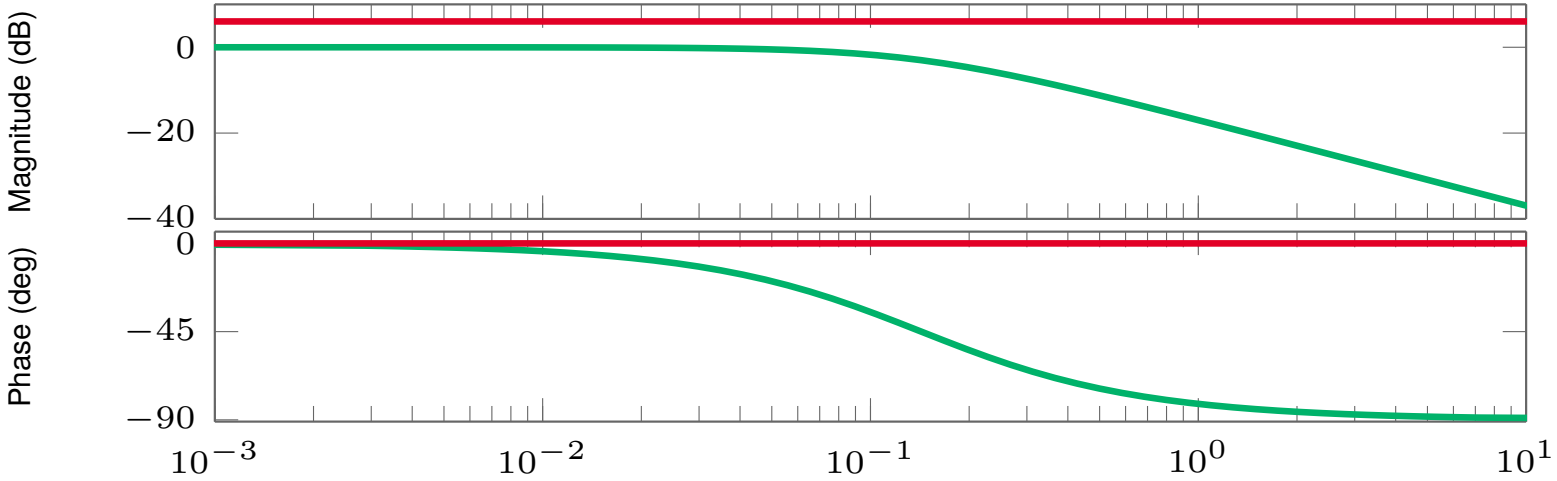
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

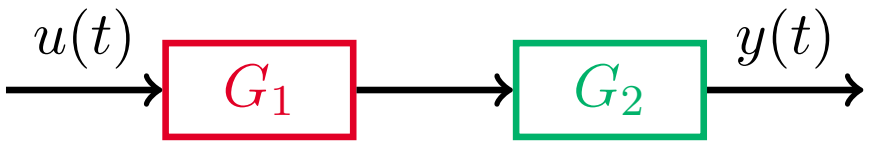
- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB$.
- **Phase shift:** $\arg(G_1(i\omega)) = \arg(2) = 0$.
- Add the curves!



per.mattsson@hig.se

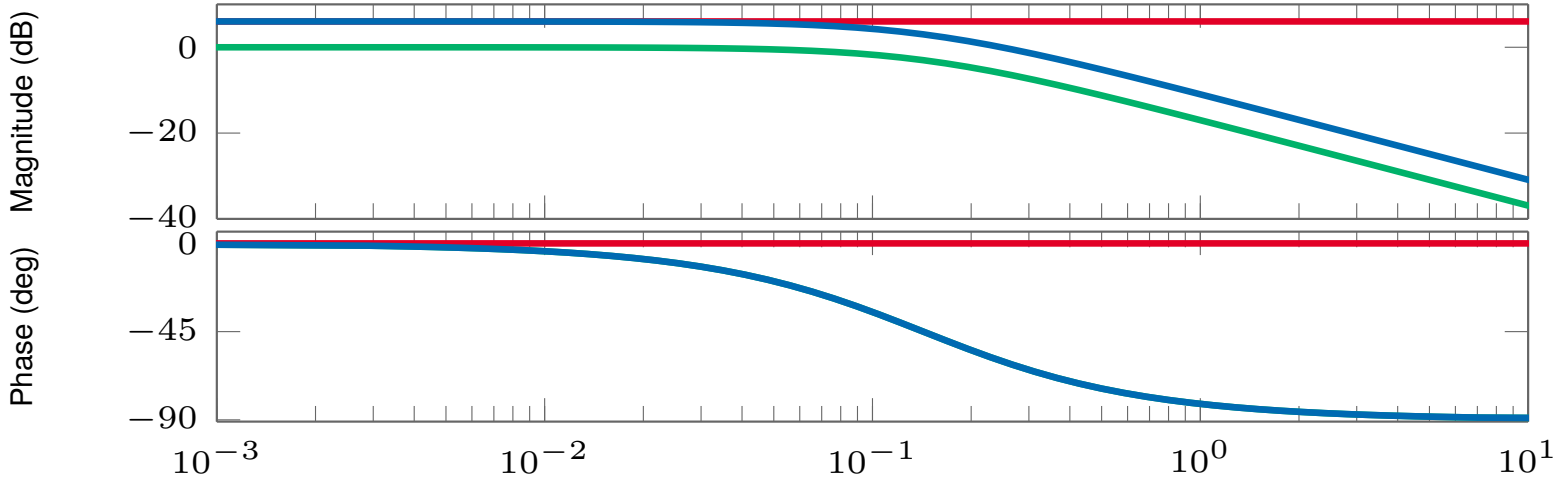
Bode plots for systems connected in series

Example



If the Bode plot of $G_2(s)$ is given by the figure, and $G_1(s) = 2$, what is the Bode plot for $G(s) = G_2(s)G_1(s)$?

- **Amplitude gain:** $|G_1(i\omega)|_{dB} = 20\log_{10}(2) \approx 6.02\,dB$.
- **Phase shift:** $\arg(G_1(i\omega)) = \arg(2) = 0$.
- Add the curves!



per.mattsson@hig.se

Nyquist plots

- Named after the Swedish born electronic engineer Harry Nyquist.
- Plot Re$[G(i\omega)]$ against Im$[G(i\omega)]$.

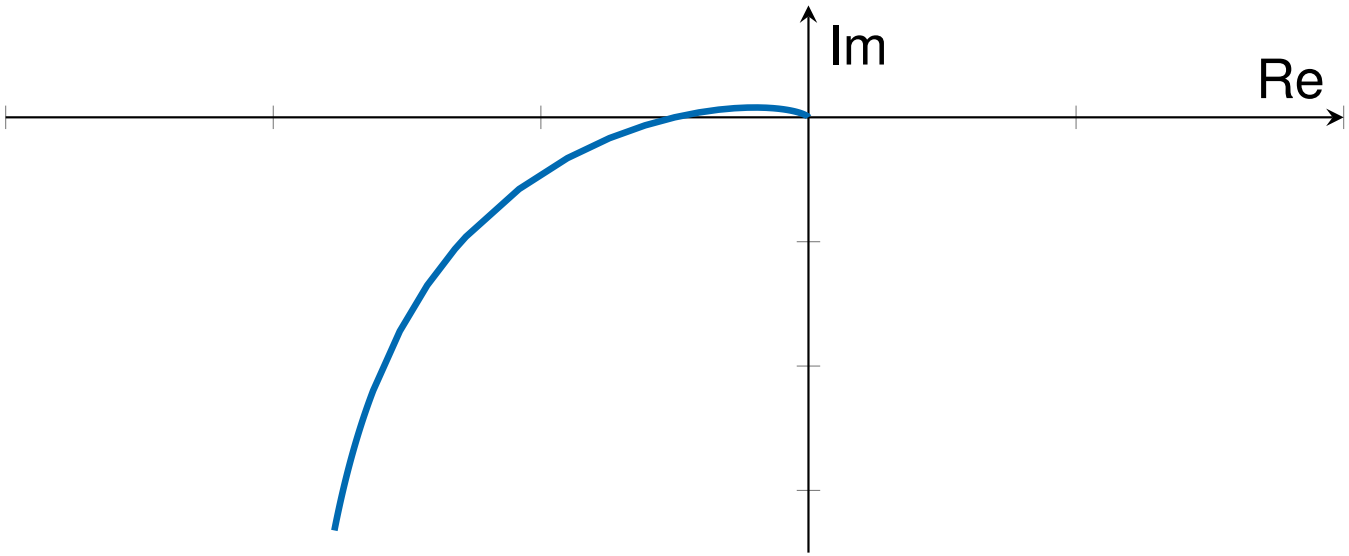


Figure: $G(s) = \frac{1}{s(1+s)^2}$.

per.mattsson@hig.se

Nyquist plots

- Named after the Swedish born electronic engineer Harry Nyquist.
- Plot Re$[G(i\omega)]$ against Im$[G(i\omega)]$.
- Can mark different frequencies.

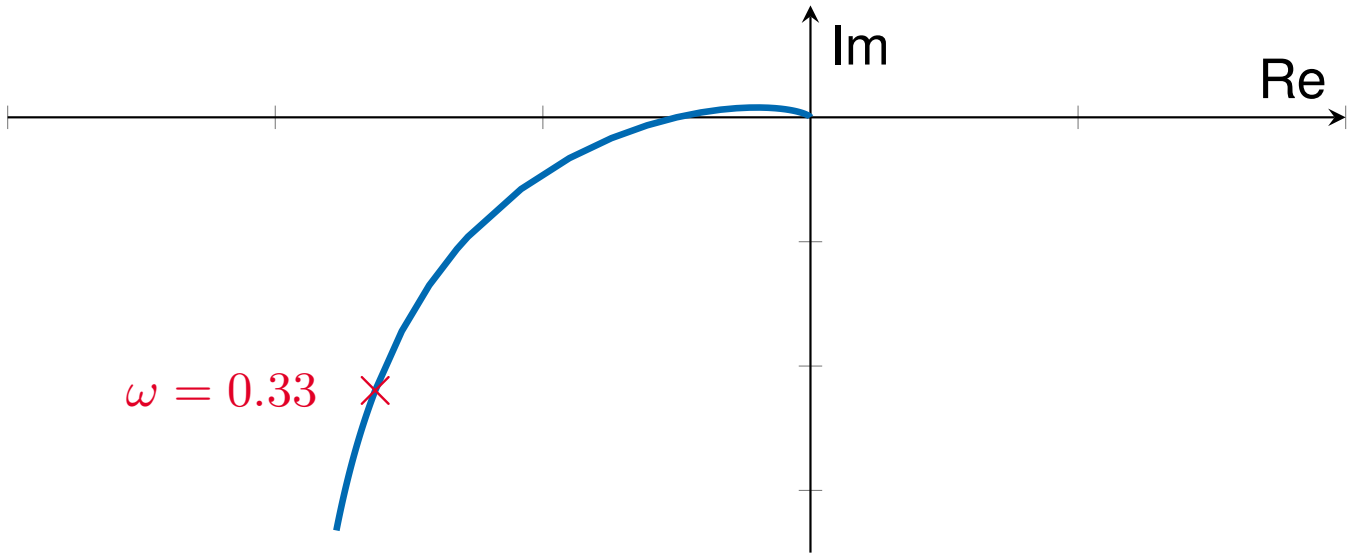


Figure: $G(s) = \frac{1}{s(1+s)^2}$.

per.mattsson@hig.se

Nyquist plots

- Named after the Swedish born electronic engineer Harry Nyquist.
- Plot Re$[G(i\omega)]$ against Im$[G(i\omega)]$.
- Can mark different frequencies.
- Can get the amplitude gain and phase shift from the plot.

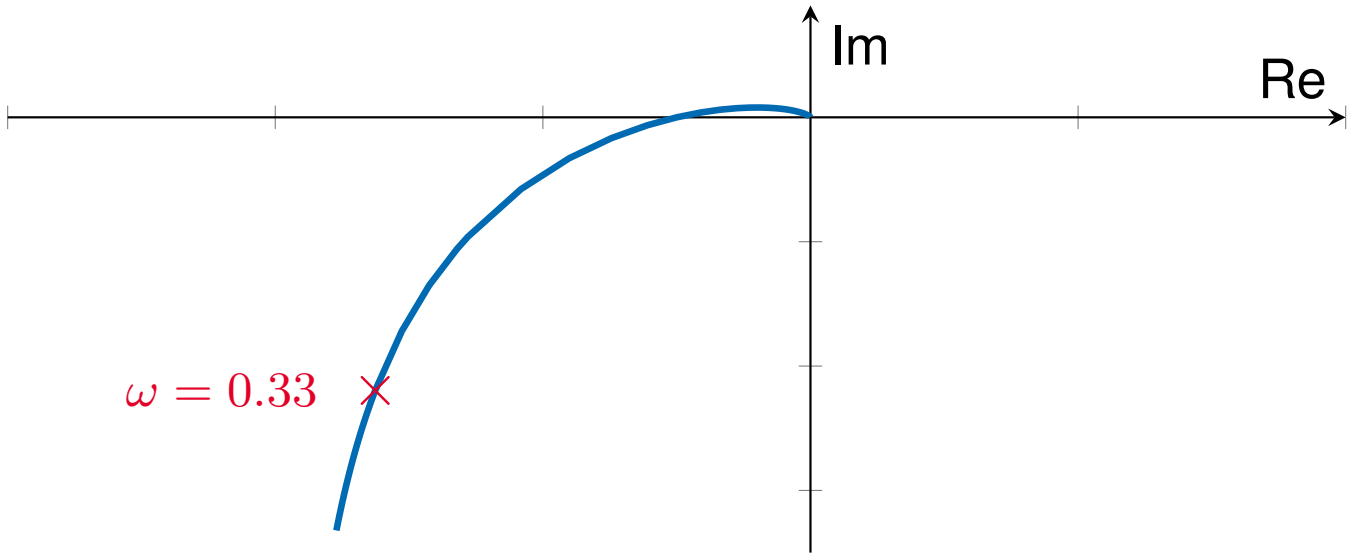


Figure: $G(s) = \frac{1}{s(1+s)^2}$.

per.mattsson@hig.se

Nyquist plots

- Named after the Swedish born electronic engineer Harry Nyquist.
- Plot Re$[G(i\omega)]$ against Im$[G(i\omega)]$.
- Can mark different frequencies.
- Can get the amplitude gain and phase shift from the plot.

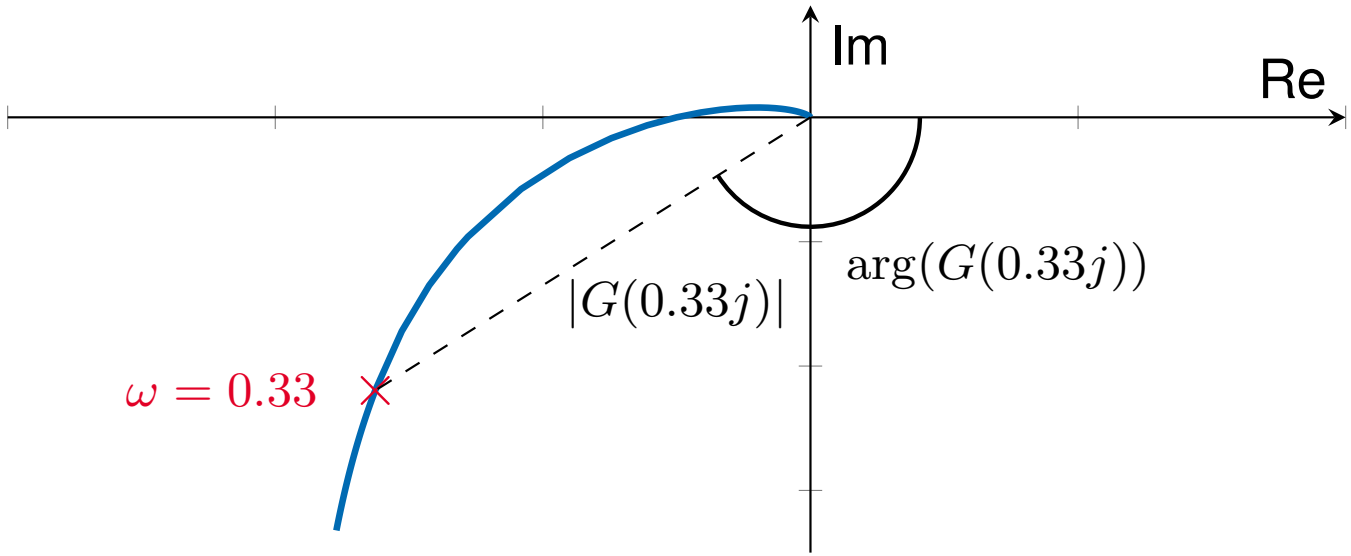


Figure: $G(s) = \frac{1}{s(1+s)^2}$.

per.mattsson@hig.se

Nyquist plots

- Named after the Swedish born electronic engineer Harry Nyquist.
- Plot Re$[G(i\omega)]$ against Im$[G(i\omega)]$.
- Can mark different frequencies.
- Can get the amplitude gain and phase shift from the plot.

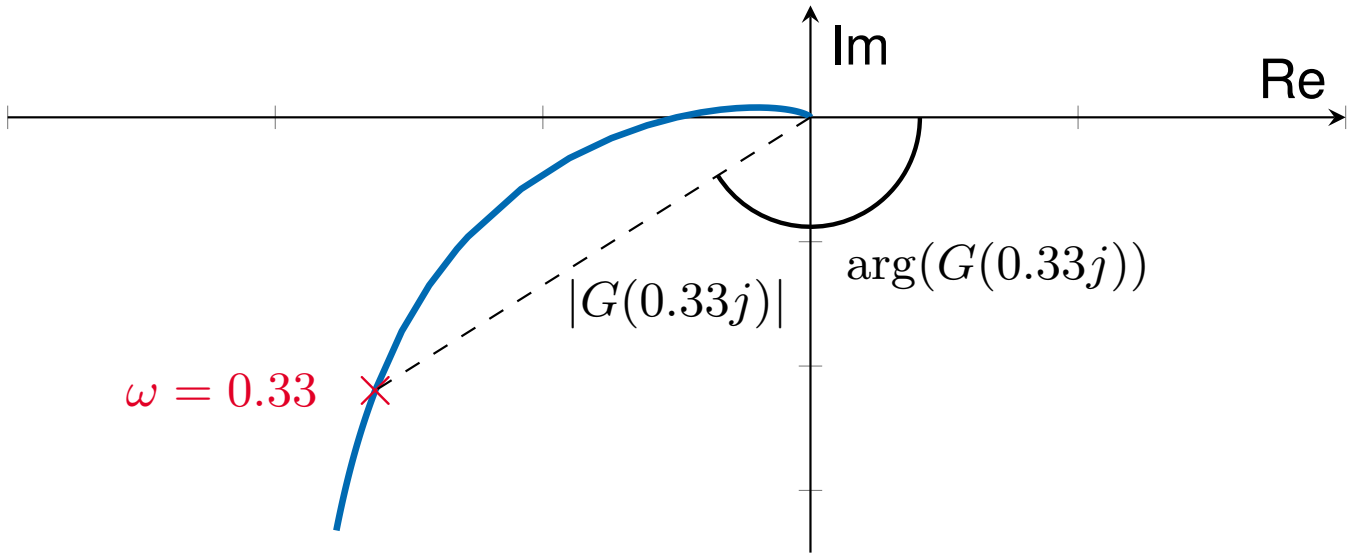


Figure: $G(s) = \frac{1}{s(1+s)^2}$.

MATLAB: `» G = tf(1, [1 2 1 0]); nyquist(G)`

per.mattsson@hig.se