HÖGSKOLAN I GÄVLE

Review of Automatic Control

Poles, stability and more

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Introduction

$$\xrightarrow{u(t)} \boxed{G} \xrightarrow{y(t)}$$

- In control, the concept of stability is very important.

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Introduction

- In control, the concept of stability is very important.
- **Bounded-input/bounded-output (BIBO):** We say that a system is BIBO-stable if a bounded input $u(t)$ results in a bounded output $y(t)$.

Poles, zeros, stability

$$\xrightarrow{u(t)} \boxed{G(s)} \xrightarrow{y(t)}$$

Consider a linear SISO-system

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}.$$

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Poles, zeros, stability

$$u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

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- **Poles:** The poles of $G(s)$ are given by the zeros of the denominator, i.e., the solutions to

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0.$$

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- Zeros: The zeros of $G(s)$ are given by the zeros of the numerator, i.e., the solutions to

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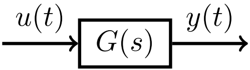
Poles, zeros, stability



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- **Stability:** $G(s)$ is stable if and only if all poles have strictly negative real part (lie in the left half-plane).

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State space models

Consider the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

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$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

- **Poles:** The poles of the system, i.e. zeros of the denominator in $G(s) = C(sI - A)^{-1}B + D$, are eigenvalues of A.

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- **Solution to the state equation:**

$$x(t) = e^{A(t-t_o)}x(t_o) + \int_{t_o}^{t} e^{A(t-\tau)}Bu(\tau)d\tau,$$

where e^{At} is the matrix exponential of At.

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  where e^{At} is the matrix exponential of At.
- If all eigenvalues of A have strictly negative real-part, then

$$e^{At} \to 0, \quad \text{as} \quad t \to \infty.$$

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The matrix exponential

- The matrix exponential e^{At} is defined as

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \cdots + \frac{1}{k!}(At)^k + \cdots$$

and can be computed using

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right](t).$$

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- If the eigenvalues of A are given by $\lambda_1, \dots, \lambda_n$, then the eigenvalues of e^{At} are given by $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$.

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- If the eigenvalues of A are given by $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of e^{At} are given by $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$.
- Note that the eigenvalues of e^{At} goes to 0 as $t \to \infty$ if and only if they have strictly negative real-part.

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Diagonal form

If A is a diagonal matrix,

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \text{ then } e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}.$$

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Example

Consider the system

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t).$$

The eigenvalues of A (and the poles of the system) are λ_1 and λ_2.

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The eigenvalues of A (and the poles of the system) are λ_1 and λ_2. If $u(t) = 0$, and the initial state is $x(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then the solution is

$$x(t) = e^{At} x(0)$$

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$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Hence, if the system is stable (all poles in the left half-plane), then $y(t) \to 0$ if $u(t) = 0$.

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The eigenvalues of A (and the poles of the system) are λ_1 and λ_2. If $u(t) = 0$, and the initial state is $x(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then the solution is

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$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Hence, if the system is stable (all poles in the left half-plane), then $y(t) \to 0$ if $u(t) = 0$. But if at least one pole is in the right half-plane, then $|y(t)| \to \infty$.

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