Review of Automatic Control

State space models

Per Mattsson

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Introduction

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$$\xrightarrow{u(t)} \boxed{G} \xrightarrow{y(t)}$$

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$$Y(s) = G(s)U(s)$$

- This form is especially useful when we study how the system reacts to different frequencies in the input.
- Another popular type of linear models are the state space models.

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State space models

- In a dynamical system, the output $y(t)$ at time t does not only depend on the current input $u(t)$, but also on $u(\tau)$, $\tau < t$.

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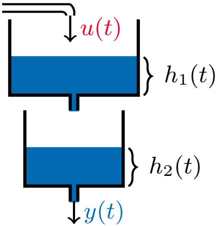

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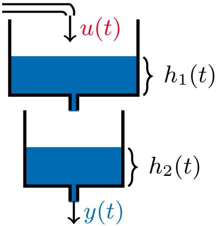

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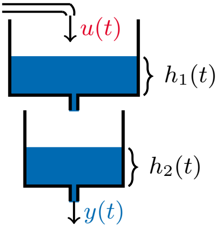

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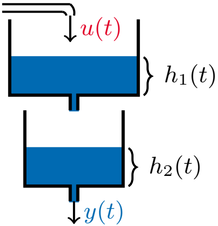

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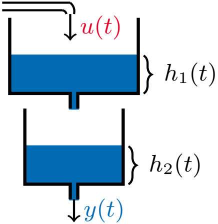
- Infinite number of ways to choose the states.



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State space form

Linear system

An LTI-system can be written on state-space form as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where $x(t)$ is the state vector, and A, B, C, D are matrices.

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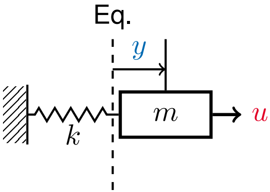

Using Newton's second law, and Hooke's law we get:

$$\ddot{y}(t) = -\frac{k}{m}y(t) + \frac{1}{m}u(t) \qquad (1)$$

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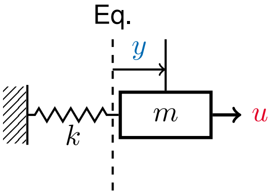

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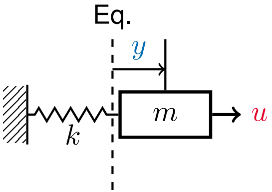

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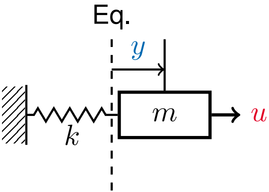

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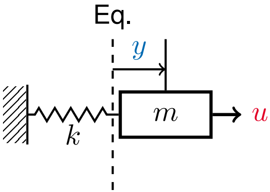

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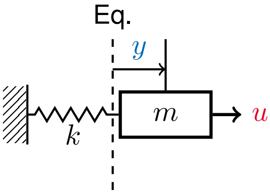

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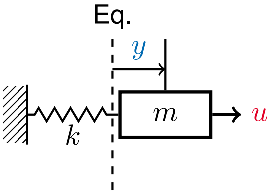

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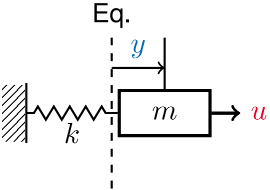

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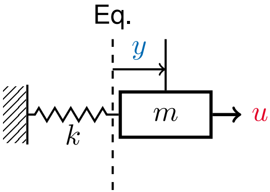

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From state space form to transfer function

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MATLAB: » `sys = ss(A,B,C,D); G = tf(sys);`

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- Canonical forms.

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Canonical forms

Observable canonical form

A SISO-system with the transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}.$$

Observable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x(t)$$

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Canonical forms

Controllable form

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Controllable canonical form:

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$$y(t) = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x(t).$$

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Some concepts

- Let n be the number of states (elements in $x(t)$)

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- **Observable** if and only if the matrix

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- **Minimal realization** if and only if both controllable and observable. For a minimal realization, it is not possible to find a state space representation with fewer states.

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