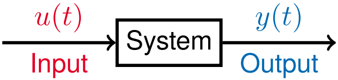
Review of Automatic Control

Transfer functions and block diagrams

Per Mattsson

per.mattsson@hig.se

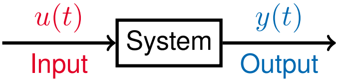
Introduction



- In order to design a controller, and analyse the closed-loop system, a **mathematical model** of the system is typically needed.

per.mattsson@hig.se

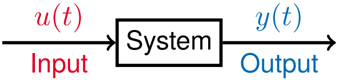
Introduction



- In order to design a controller, and analyse the closed-loop system, a **mathematical model** of the system is typically needed.
- Linear models are the most popular models, and they can often approximate the true system quite well.

per.mattsson@hig.se

Introduction



- In order to design a controller, and analyse the closed-loop system, a **mathematical model** of the system is typically needed.
- Linear models are the most popular models, and they can often approximate the true system quite well.
- **Linear model:** If the input $u_1(t)$ gives the output $y_1(t)$, and the input $u_2(t)$ gives the output $y_2(t)$, then

$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t), \text{ gives } y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

per.mattsson@hig.se

Differential equations

One common way to represent a linear dynamical system is by the use of differential equations.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

per.mattsson@hig.se

Differential equations

One common way to represent a linear dynamical system is by the use of differential equations.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

- We also write the time derivative using the dot-notation:

$$\frac{dy}{dt} = \dot{y}(t), \quad \frac{d^2 y}{dt^2} = \ddot{y}(t).$$

per.mattsson@hig.se

Differential equations

One common way to represent a linear dynamical system is by the use of differential equations.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

- We also write the time derivative using the dot-notation:

$$\frac{dy}{dt} = \dot{y}(t), \quad \frac{d^2 y}{dt^2} = \ddot{y}(t).$$

- Working with high-order differential equations directly is often inconvenient, and therefore the Laplace-transform is often used in control theory.

per.mattsson@hig.se

The Laplace transform

- **Notation:**

$$f(t) \xleftrightarrow{\mathcal{L}} \mathcal{L}\left[f(t)\right] = F(s)$$

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

The Laplace transform

- **Notation:**

$$f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \mathcal{L}\left[f(t)\right] = F(s)$$

- **Definition:**

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt, \quad s \in \mathbb{C}$$

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

The Laplace transform

- **Notation:**

$$f(t) \xleftrightarrow{\mathcal{L}} \mathcal{L}\left[f(t)\right] = F(s)$$

- **Definition:**

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt, \quad s \in \mathbb{C}$$

- **Inverse transform:**

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} e^{st}F(s)ds.$$

per.mattsson@hig.se

Important properties of the Laplace transform

Linear $$\alpha x(t) + \beta z(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \alpha X(s) + \beta Z(s)$$

per.mattsson@hig.se

Important properties of the Laplace transform

Linear $$\alpha x(t) + \beta z(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \alpha X(s) + \beta Z(s)$$

Derivatives

(if at rest at $t = 0$)

$$\frac{df}{dt} \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad sF(s)$$

$$\frac{d^n f}{dt^n} \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad s^n F(s)$$

per.mattsson@hig.se

Important properties of the Laplace transform

Linear	$\alpha x(t) + \beta z(t)$	$\stackrel{\mathcal{L}}{\longleftrightarrow}$	$\alpha X(s) + \beta Z(s)$
Derivatives	$\dfrac{df}{dt}$	$\stackrel{\mathcal{L}}{\longleftrightarrow}$	$sF(s)$
(if at rest at $t = 0$)	$\dfrac{d^n f}{dt^n}$	$\stackrel{\mathcal{L}}{\longleftrightarrow}$	$s^n F(s)$
Integral	$\int_0^t f(t)dt$	$\stackrel{\mathcal{L}}{\longleftrightarrow}$	$\dfrac{1}{s}F(s)$

per.mattsson@hig.se

Important properties of the Laplace transform

Linear	$\alpha x(t) + \beta z(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\alpha X(s) + \beta Z(s)$
Derivatives (if at rest at $t = 0$)	$\dfrac{df}{dt}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$sF(s)$
	$\dfrac{d^n f}{dt^n}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$s^n F(s)$
Integral	$\displaystyle\int_0^t f(t)dt$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\dfrac{1}{s}F(s)$
Final value*	$\displaystyle\lim_{t\to\infty} f(t)$	$=$	$\displaystyle\lim_{s\to 0} sF(s)$

**If the limit on the left-hand side exists.*

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$.

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) Y(s) = (b_1 s^{n-1} + \cdots + b_{n-1} s + b_n) U(s),$$

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) Y(s) = (b_1 s^{n-1} + \cdots + b_{n-1} s + b_n) U(s),$$

or equivalently

$$Y(s) = \underbrace{\frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}_{G(s)} U(s)$$

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) Y(s) = (b_1 s^{n-1} + \cdots + b_{n-1} s + b_n) U(s),$$

or equivalently

$$Y(s) = \underbrace{\frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}_{G(s)} U(s) = G(s) U(s)$$

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy}{dt}+a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}}+\cdots+a_{n-1}\frac{du}{dt}+a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

$$(s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n)Y(s) = (b_1 s^{n-1}+\cdots+b_{n-1}s+b_n)U(s),$$

or equivalently

$$Y(s) = \underbrace{\frac{b_1 s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}}_{G(s)}U(s) = G(s)U(s)$$

We call $G(s) = \frac{Y(s)}{U(s)}$ the **transfer function**, since it shows how the input is transferred to the output.

per.mattsson@hig.se

Transfer functions

Assume that the linear system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_{n-1} \frac{du}{dt} + a_n u$$

is at rest at $t = 0$. Taking the Laplace transform we get

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) Y(s) = (b_1 s^{n-1} + \cdots + b_{n-1} s + b_n) U(s),$$

or equivalently

$$Y(s) = \underbrace{\frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}_{G(s)} U(s) = G(s) U(s)$$

We call $G(s) = \frac{Y(s)}{U(s)}$ the **transfer function**, since it shows how the input is transferred to the output.

MATLAB: » `G = tf([b1 b2 ...  bn], [1 a1 ...  an])`

per.mattsson@hig.se

The impulse response

In a linear system, the output is a weighted integral of past inputs:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau.$$

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

The impulse response

In a linear system, the output is a weighted integral of past inputs:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau.$$

- Note that if $u(t) = \delta(t)$, then $y(t) = g(t)$.

per.mattsson@hig.se

The impulse response

In a linear system, the output is a weighted integral of past inputs:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau.$$

- Note that if $u(t) = \delta(t)$, then $y(t) = g(t)$.
- Therefore $g(t)$ is called the impulse response.

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

The impulse response

In a linear system, the output is a weighted integral of past inputs:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau.$$

- Note that if $u(t) = \delta(t)$, then $y(t) = g(t)$.
- Therefore $g(t)$ is called the impulse response.
- If we know $g(t)$ then we in principle know how the system reacts to any input.

per.mattsson@hig.se

The impulse response

In a linear system, the output is a weighted integral of past inputs:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau.$$

- Note that if $u(t) = \delta(t)$, then $y(t) = g(t)$.
- Therefore $g(t)$ is called the impulse response.
- If we know $g(t)$ then we in principle know how the system reacts to any input.
- The **transfer function** is the Laplace transform of the impulse response:

$$G(s) = \mathcal{L}[g] = \int_0^\infty e^{-st} g(t)dt.$$

per.mattsson@hig.se

Example

Consider the linear differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 6y(t) = u(t).$$

per.mattsson@hig.se

Example

Consider the linear differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 6y(t) = u(t).$$

Taking the Laplace transform of both sides (assuming system at rest at $t = 0$), we get

$$(s^2 + 3s + 6)Y(s) = U(s)$$

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

Example

Consider the linear differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 6y(t) = u(t).$$

Taking the Laplace transform of both sides (assuming system at rest at $t = 0$), we get

$$(s^2 + 3s + 6)Y(s) = U(s),$$

so the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 6}.$$

per.mattsson@hig.se

Block diagrams

- We often illustrate a linear system with input $x(t)$ and output $y(t)$ in a block diagram:

$$x(t) \longrightarrow \boxed{G} \longrightarrow y(t)$$

per.mattsson@hig.se

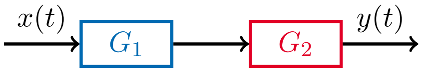
Block diagrams

- We often illustrate a linear system with input $x(t)$ and output $y(t)$ in a block diagram:

- Block diagram can be very useful for illustrating how subsystems interact.

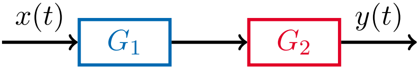
per.mattsson@hig.se

Example: Connected in series



per.mattsson@hig.se

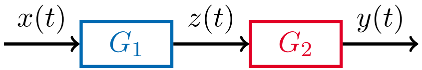
Example: Connected in series



Transfer function from x to y:

per.mattsson@hig.se

Example: Connected in series

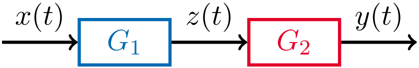


Transfer function from x to y:

Introduce the signal z.

per.mattsson@hig.se

Example: Connected in series



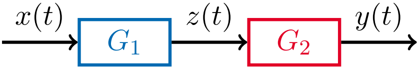
Transfer function from x to y:

Introduce the signal z.

$$Z(s) = G_1(s)X(s)$$

per.mattsson@hig.se

Example: Connected in series



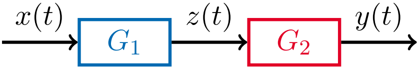
Transfer function from x to y:

Introduce the signal z.

$$Z(s) = G_1(s)X(s)$$
$$Y(s) = G_2(s)Z(s)$$

per.mattsson@hig.se

Example: Connected in series



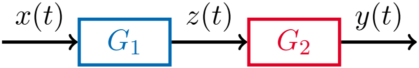
Transfer function from x to y:

Introduce the signal z.

$$
\begin{aligned}
Z(s) &= G_1(s)X(s) \\
Y(s) &= G_2(s)Z(s) = G_2(s)G_1(s)X(s).
\end{aligned}
$$

per.mattsson@hig.se

Example: Connected in series



Transfer function from x to y:

Introduce the signal z.

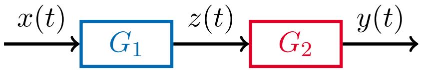
$$
\begin{aligned}
Z(s) &= G_1(s)X(s) \\
Y(s) &= G_2(s)Z(s) = G_2(s)G_1(s)X(s).
\end{aligned}
$$

Hence,

$$G(s) = \frac{Y(s)}{X(s)} =$$

per.mattsson@hig.se

Example: Connected in series



Transfer function from x to y:

Introduce the signal z.

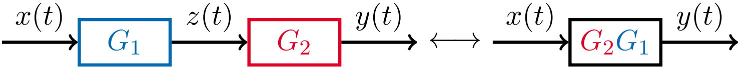
$$Z(s) = G_1(s)X(s)$$
$$Y(s) = G_2(s)Z(s) = G_2(s)G_1(s)X(s).$$

Hence,

$$G(s) = \frac{Y(s)}{X(s)} = G_2(s)G_1(s).$$

per.mattsson@hig.se

Example: Connected in series



Transfer function from x to y:

Introduce the signal z.

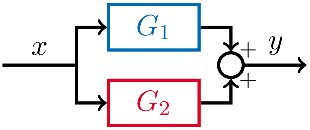
$$
\begin{aligned}
Z(s) &= G_1(s)X(s) \\
Y(s) &= G_2(s)Z(s) = G_2(s)G_1(s)X(s).
\end{aligned}
$$

Hence,

$$G(s) = \frac{Y(s)}{X(s)} = G_2(s)G_1(s).$$

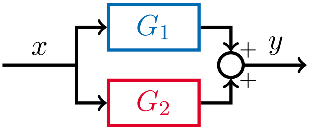
per.mattsson@hig.se

Example: Connected in parallel



per.mattsson@hig.se

Example: Connected in parallel

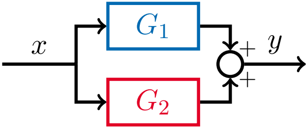


Transfer function from X to Y:

$$Y(s) =$$

per.mattsson@hig.se

Example: Connected in parallel

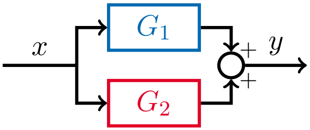


Transfer function from X to Y:

$$Y(s) = G_1(s)X(s) + G_2(s)X(s)$$

per.mattsson@hig.se

Example: Connected in parallel

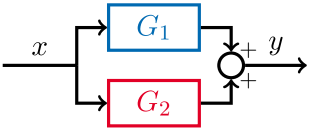


Transfer function from X to Y:

$$Y(s) = G_1(s)X(s) + G_2(s)X(s) = (G_1(s) + G_2(s))\,X(s).$$

per.mattsson@hig.se

Example: Connected in parallel



Transfer function from X to Y:

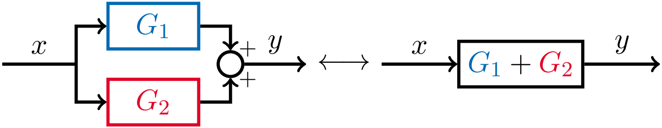
$$Y(s) = G_1(s)X(s) + G_2(s)X(s) = (G_1(s) + G_2(s))\,X(s).$$

Hence,

$$G(s) = \frac{Y(s)}{X(s)} = G_1(s) + G_2(s).$$

per.mattsson@hig.se

Example: Connected in parallel



Transfer function from X to Y:

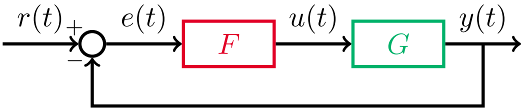
$$Y(s) = G_1(s)X(s) + G_2(s)X(s) = (G_1(s) + G_2(s))\,X(s).$$

Hence,

$$G(s) = \frac{Y(s)}{X(s)} = G_1(s) + G_2(s).$$

per.mattsson@hig.se

Example: Negative feedback

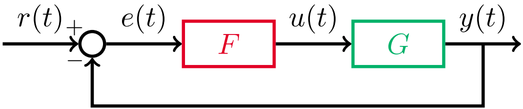


Transfer function from $r(t)$ to $y(t)$:

$$Y(s) = G(s)U(s)$$

per.mattsson@hig.se

Example: Negative feedback

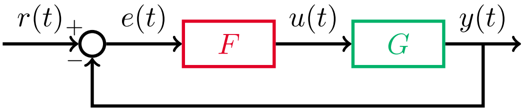


Transfer function from $r(t)$ to $y(t)$:

$$Y(s) = G(s)U(s) = G(s)F(s)E(s)$$

per.mattsson@hig.se

Example: Negative feedback

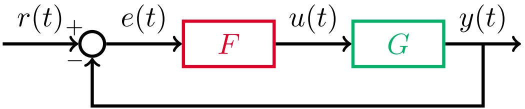


Transfer function from $r(t)$ to $y(t)$:

$$
\begin{aligned}
Y(s) &= G(s)U(s) = G(s)F(s)E(s) \\
&= G(s)F(s)(R(s) - Y(s)).
\end{aligned}
$$

per.mattsson@hig.se

Example: Negative feedback



Transfer function from $r(t)$ to $y(t)$:

$$
\begin{aligned}
Y(s) &= G(s)U(s) = G(s)F(s)E(s) \\
&= G(s)F(s)(R(s) - Y(s)).
\end{aligned}
$$

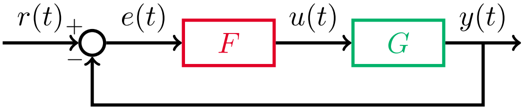
Hence,

$$(1 + F(s)G(s))Y(s) = F(s)G(s)R(s)$$

per.mattsson@hig.se

HÖGSKOLAN I GÄVLE

Example: Negative feedback



Transfer function from $r(t)$ to $y(t)$:

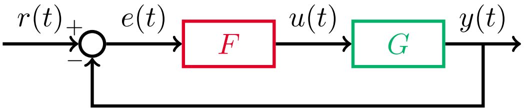
$$\begin{aligned} Y(s) &= G(s)U(s) = G(s)F(s)E(s) \\ &= G(s)F(s)(R(s) - Y(s)). \end{aligned}$$

Hence,

$$\begin{aligned} (1 + F(s)G(s))Y(s) &= F(s)G(s)R(s) \iff \\ Y(s) &= \frac{F(s)G(s)}{1 + F(s)G(s)} R(s). \end{aligned}$$

per.mattsson@hig.se

Example: Negative feedback



Transfer function from $r(t)$ to $y(t)$:

$$\begin{aligned} Y(s) &= G(s)U(s) = G(s)F(s)E(s) \\ &= G(s)F(s)(R(s) - Y(s)). \end{aligned}$$

Hence,

$$\begin{aligned} (1 + F(s)G(s))Y(s) &= F(s)G(s)R(s) \iff \\ Y(s) &= \frac{F(s)G(s)}{1 + F(s)G(s)} R(s). \end{aligned}$$

So, the transfer function from $r(t)$ to $y(t)$ is

$$G_c(s) = \frac{Y(s)}{R(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)}.$$

per.mattsson@hig.se